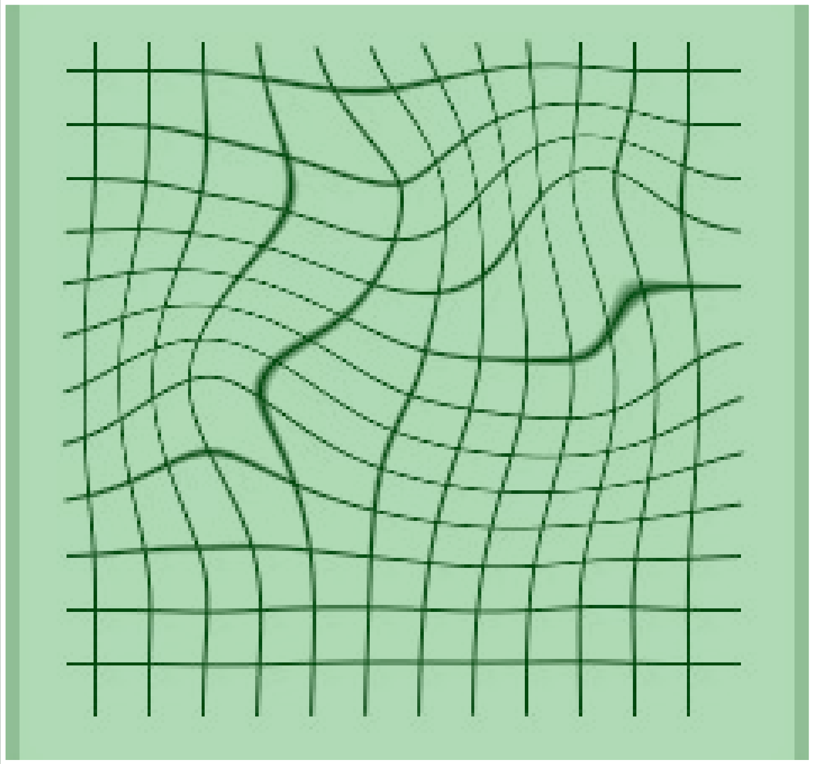
Nonperturbative quantum field theory on a real time lattice

Szabolcs Borsanyi, Heidelberg



In collaboration with

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Real time simulations?

DAYDREAMS...

- spectral function, decay rates
- real time response → nonequilibrium
- nonequilbirum field theory without
  - Rayleigh-Jeans problem (classical)
  - Gauge dependence problems
- could we simulate a heavy ion collision?

(short period of time: a few fm/c would be enough)

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→stochastic quantization techniques

Stochastic quantization

Parisi, Wu 1981

$$\partial_\vartheta \phi(x,\vartheta) = -\frac{\delta S_E[\phi]}{\delta\phi(x,\vartheta)} + \eta(x,\vartheta) \qquad \langle \eta(x_1,\vartheta_1)\eta(x_2,\vartheta_2)\rangle = 2\delta(\vartheta_1-\vartheta_2)\delta^{(4)}(x_1-x_2)$$

$$\phi(\vartheta+\varepsilon) = \phi(\vartheta) - \varepsilon\frac{\partial S_E}{\partial\phi} + \sqrt{2\varepsilon}\,\xi \qquad \langle\xi^2\rangle = 1 \text{ random number}$$

$$\frac{\partial\phi(x,t)}{\partial t} = i\,\frac{\delta S[\phi]}{\delta\phi(x,t)} + \eta(x,t)$$

Huffel, Rumpf 1984
Gozzi 1984
Klauder 1984

$$\begin{aligned}
\varphi'_R(t,\mathbf{x}) &= \varphi_R(t,\mathbf{x}) - \epsilon\, I_\xi(\varphi_R,\varphi_I;t,\mathbf{x}) + \sqrt{\epsilon}\,\eta_R(t,\mathbf{x}) \\
\varphi'_I(t,\mathbf{x}) &= \varphi_I(t,\mathbf{x}) + \epsilon\, R_\xi(\varphi_R,\varphi_I;t,\mathbf{x}) + \sqrt{\epsilon}\,\eta_I(t,\mathbf{x})
\end{aligned}$$

$$\begin{aligned}
R_\xi(\varphi_R,\varphi_I;t,\mathbf{x}) &\equiv \mathrm{Re}\left(\left.\frac{\delta S_\xi[\varphi]}{\delta\varphi(t,\mathbf{x})}\right|_{\varphi=\varphi_R+i\varphi_I}\right) \\
I_\xi(\varphi_R,\varphi_I;t,\mathbf{x}) &\equiv \mathrm{Im}\left(\left.\frac{\delta S_\xi[\varphi]}{\delta\varphi(t,\mathbf{x})}\right|_{\varphi=\varphi_R+i\varphi_I}\right)
\end{aligned}$$

Real Fokker-Planck equation

$$\frac{\partial P[\varphi_R,\varphi_I]}{\partial\vartheta} = \int d^4x\left[\frac{\delta(PI)}{\delta\varphi_R} - \frac{\delta(PR)}{\delta\varphi_I} + \frac{\delta^2 P}{\delta\varphi_R^2}\right]$$

Reweighting vs. Stochastic quantization

Reweighting

$$\int D\Phi e^{iS_R[\Phi]-S_I[\Phi]}\Phi(t_1)\Phi(t_2) = \int D\Phi e^{-S_I[\phi]}\left[e^{iS_R[\Phi]}\Phi(t_1)\Phi(t_2)\right]$$

Action in Minkowski time: no importance sampling.

Cost of simulation: ~ exp(volume)

Stochastic quantization *(in real time)*

Analytical continuation of the distribution!
Strict argument for convergence in free theory.
Thermalization time? Convergence? Precision test?

Performance compared to stochastic quantization

Callaway at al. (1985)

The real field becomes complex

$$\langle \mathcal{O} \rangle_\vartheta = \int [d\varphi_R][d\varphi_I] \mathcal{O}(\varphi_R + i\varphi_I) P(\varphi_R, \varphi_I, \vartheta)$$

$$\langle \mathcal{O} \rangle_\vartheta = \int [d\varphi_R][d\varphi_I] \mathcal{O}(\varphi_R) P_{\text{eff}}(\varphi_R, \vartheta)$$

$$P_{\text{eff}}(\varphi_R, \vartheta) = \int [d\varphi_I] P(\varphi_R - i\varphi_I, \varphi_I, \vartheta)$$

$$P_{\text{eff}}(\varphi_R, \vartheta) \to e^{iS[\varphi_R]}$$

Lattice in Minkowski space-time:

needs a small regulator $m^2 - i\epsilon$

boudary conditions?

initial conditions?

$\rightarrow$

Real time observables:

$$\text{Tr}\hat{\rho}\hat{\phi}e^{i\hat{H}t}\hat{\phi}e^{-i\hat{H}t}$$

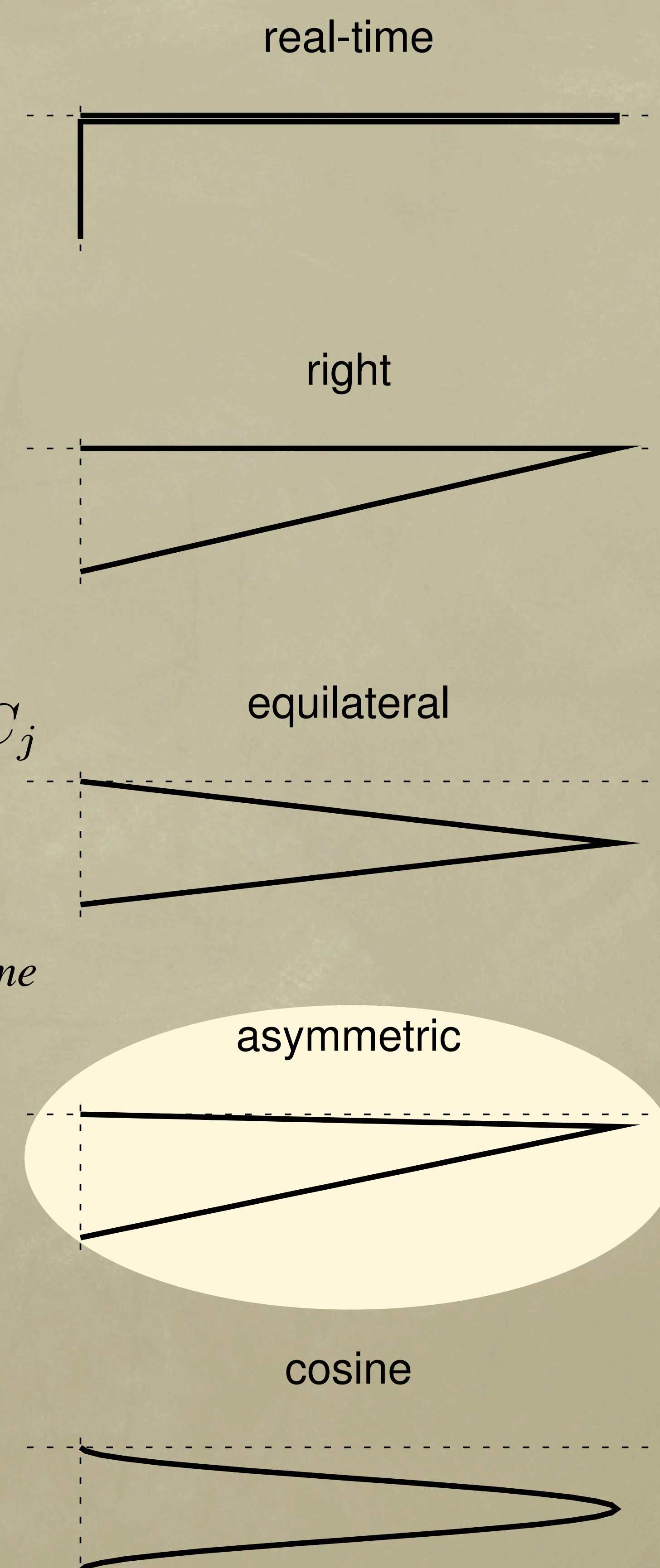
Real time equilibrium:

$$\text{Tr}e^{-\beta\hat{H}}\hat{\phi}e^{iHt}\hat{\phi}e^{-iHt}$$

<u>Contour?</u>

CTP → Complex Time Path

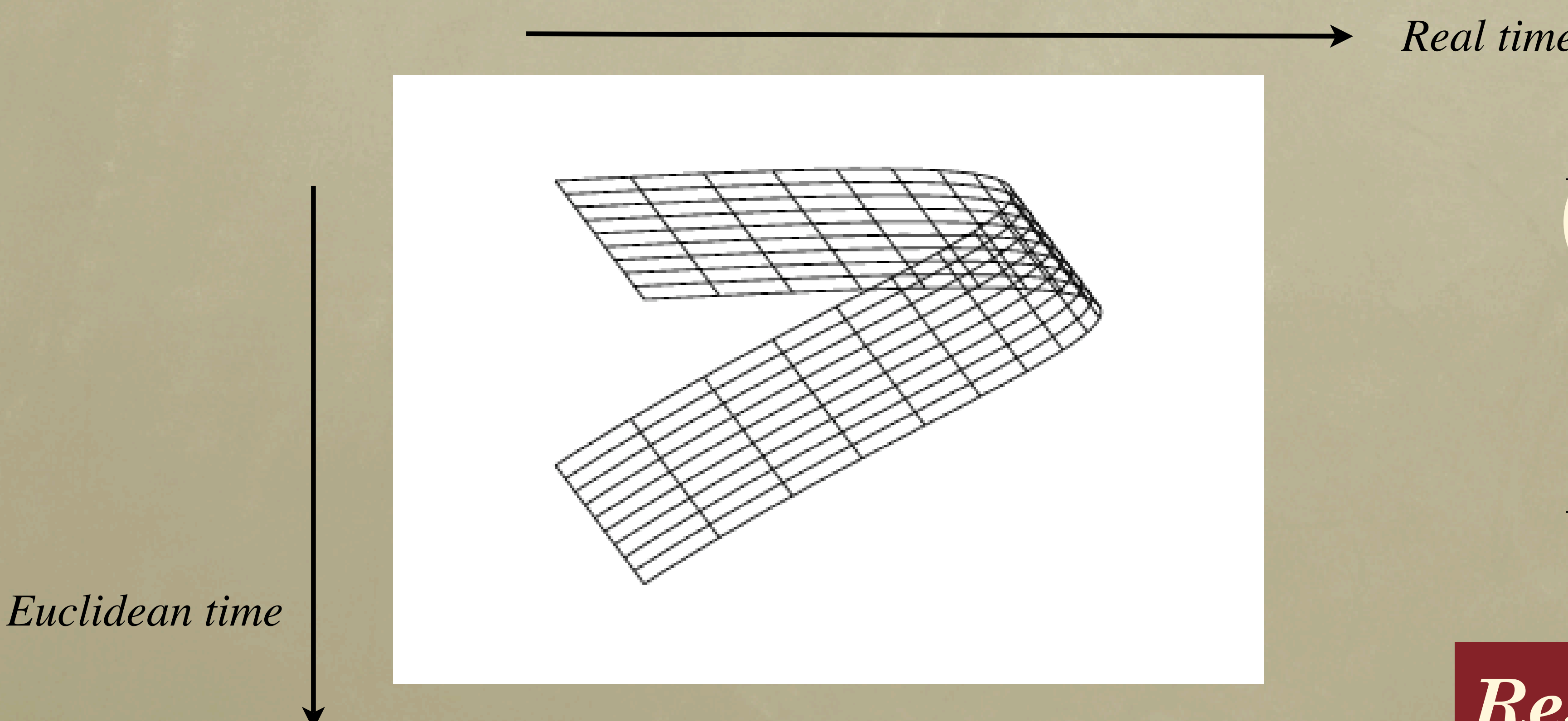
$$-iS = -\frac{i}{2}\sum_j \frac{(\phi_{j+1}-\phi_j)^2}{C_{j+1}-C_j} + \frac{i}{2}\sum_j (C_{j+1}-C_j)\,(V[\phi(C_{j+1}] + V[\phi(C_j)])$$



Langevin equation:

$$\frac{\partial \phi(C_j)}{\partial \vartheta} = i\frac{\partial S}{\partial \phi(C_j)} + \eta_j(\vartheta)$$

Contour points: C_j



Respects gauge symmetry!

Harmonic Oscillator:

Klauder 1984, Nakazato&Yamanaka 1985

$$-iS[\phi] = \frac{1}{2}\phi_i M_{ij}\phi_j$$

$\phi_j \equiv \phi(C_j)$ M: complex symmetric matrix

$$\partial_\vartheta \phi_i = -M_{ij}\phi_j + \eta_i$$

$$\phi_i = \sum_a z_a \psi_a^i$$

Eigensystem: $\vec{\psi}_a$ (orthogonal), eigenvalues λ_a (complex)

$$\boxed{\partial_\vartheta z_a = -\lambda_a z_a + {\eta'}_a}$$

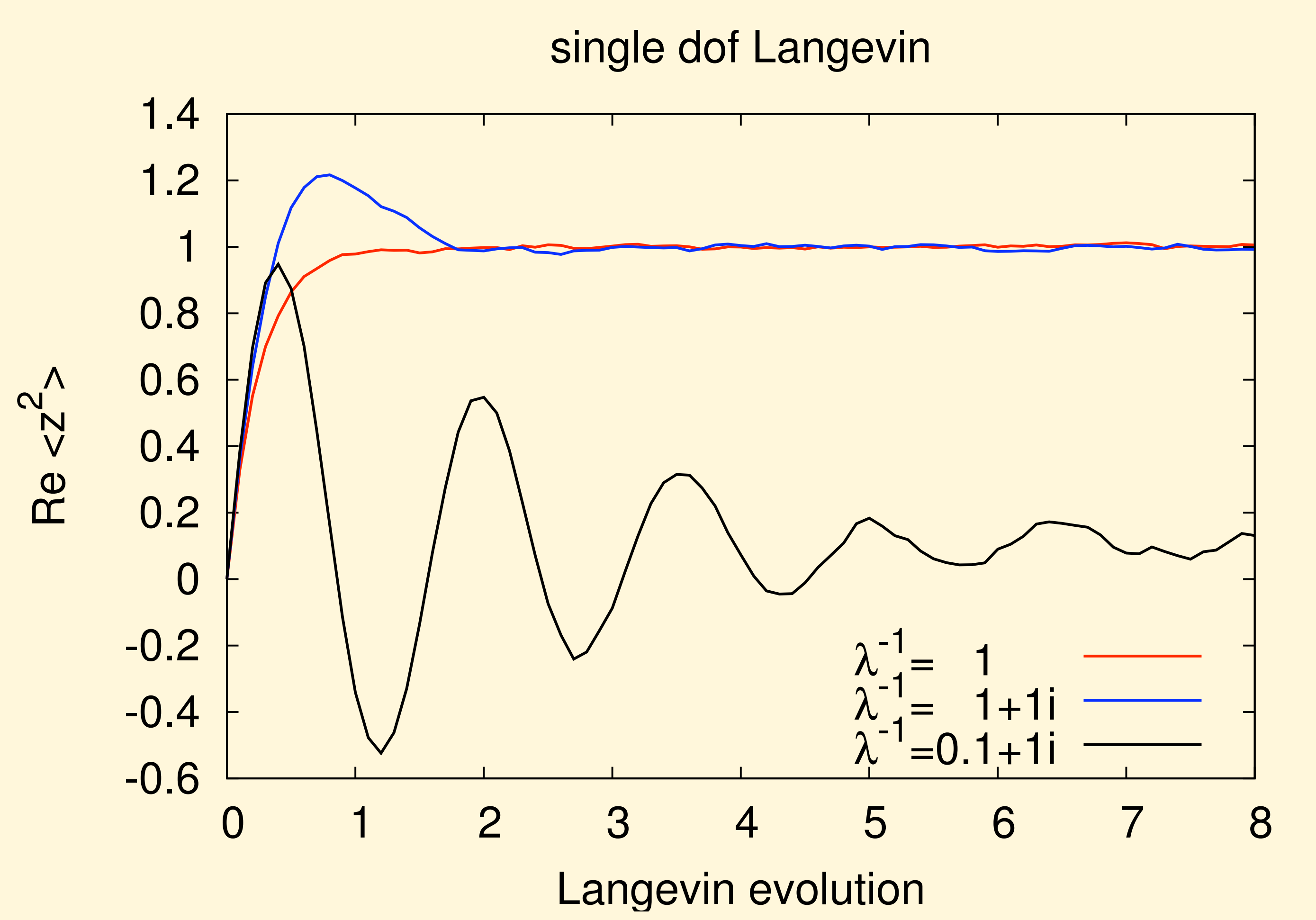
Eigenvalues depend on the contour shape.
The tilt of the contour acts as regulator.

Converges if $\mathrm{Re}\,\lambda > 0$

$$\mathrm{Re} < z^2 > \to \mathrm{Re} 1/\lambda$$
$$\mathrm{Im} < z^2 > \to \mathrm{Im} 1/\lambda$$

$$< z^2 > -1/\lambda \sim e^{-2\lambda t}$$

$$\langle z^4 \rangle_0 = 3\langle z^2 \rangle_0$$



Complex eigenvalue:

damped oscillation around the limit

Field theory: summing over modes

"Prethermalization":

accelerates convergence

Free field: analytically accessable

Nakazato&Yamanaka 1985

$$\dot\phi_R(k,t) = -q\phi_I(k,t) - \varepsilon\phi_R(k,t) + \eta(k,t),$$

$$\dot\phi_I(k,t) = q\phi_R(k,t) - \varepsilon\phi_I(k,t),$$

ε small real part of the eigenvalue $\lambda_k = \varepsilon - iq$

FP Hamiltonian

$$\frac{\partial}{\partial t}P[\phi;t] = H[\phi]P[\phi;t],$$

$$H[\phi] = \int d^4k \left\{ \frac{\delta}{\delta\phi_R(k)}\frac{\delta}{\delta\phi_R(-k)} + \frac{\delta}{\delta\phi_R(k)}[q\phi_I(k) + \varepsilon\phi_R(k)] + \frac{\delta}{\delta\phi_I(k)}[-q\phi_R(k) + \varepsilon\phi_I(k)] \right\}$$

$$\begin{aligned} P_{eq}[\phi] &= \lim_{t\to\infty} P[\phi;t] \\ &= N'\exp[-\varepsilon\int d^4k\,\{|\phi_R(k)|^2 + [1+\varepsilon^2/(\varepsilon^2+q^2)]|\phi_I(k)|^2 \\ &\qquad -2(\varepsilon/q)\phi_R(k)\phi_I(-k)\}]. \end{aligned}$$

$$P_{\text{eff}}(\varphi_R,\vartheta) = \int [d\varphi_I] P(\varphi_R - i\varphi_I, \varphi_I, \vartheta)$$

$$\begin{aligned} &\langle \phi(k)\phi(k') \rangle \\ &= \langle \phi_R(k)\phi_R(k') - \phi_I(k)\phi_I(k') \rangle + i\langle \phi_R(k)\phi_I(k') + \phi_I(k)\phi_R(k') \rangle \\ &= \delta^4(k+k')\left(\frac{\varepsilon}{q^2+\varepsilon^2} + i\frac{q}{q^2+\varepsilon^2}\right) = \delta^4(k+k')\frac{i}{q+i\varepsilon} = \delta^4(k+k')\frac{i}{k^2-m^2+i\varepsilon} \end{aligned}$$

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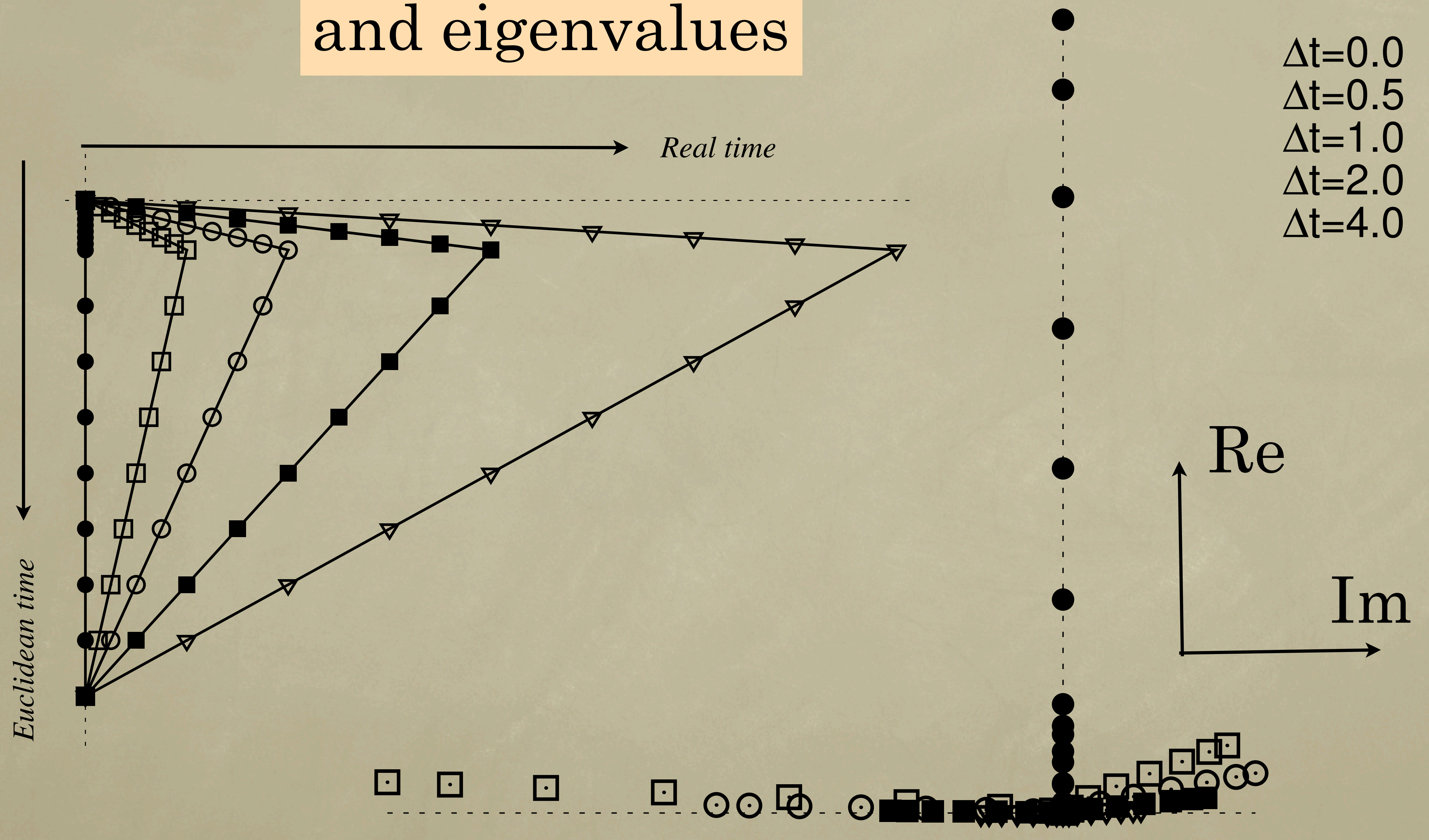
$\langle \phi(k)\phi(k') \rangle$

Perturbative progagator:

ψ^a_x: eigenfunctions of $G_0^{-1}(x,y)$

$$\langle\phi(x)\phi(y)\rangle_0 = \langle z^a z^b\rangle_0 \psi^a_x\psi^b_y = \frac{\delta^{ab}}{\lambda^a}\psi^a_x\psi^b_y = \sum_a \frac{1}{\lambda^a}\psi^a_x\psi^a_y \qquad = \left[G_0^{-1}(x,y)\right]^{-1}$$

Contour shapes and eigenvalues



The real part of the eigenvalues are always positive if there is a tilt, no matter how small

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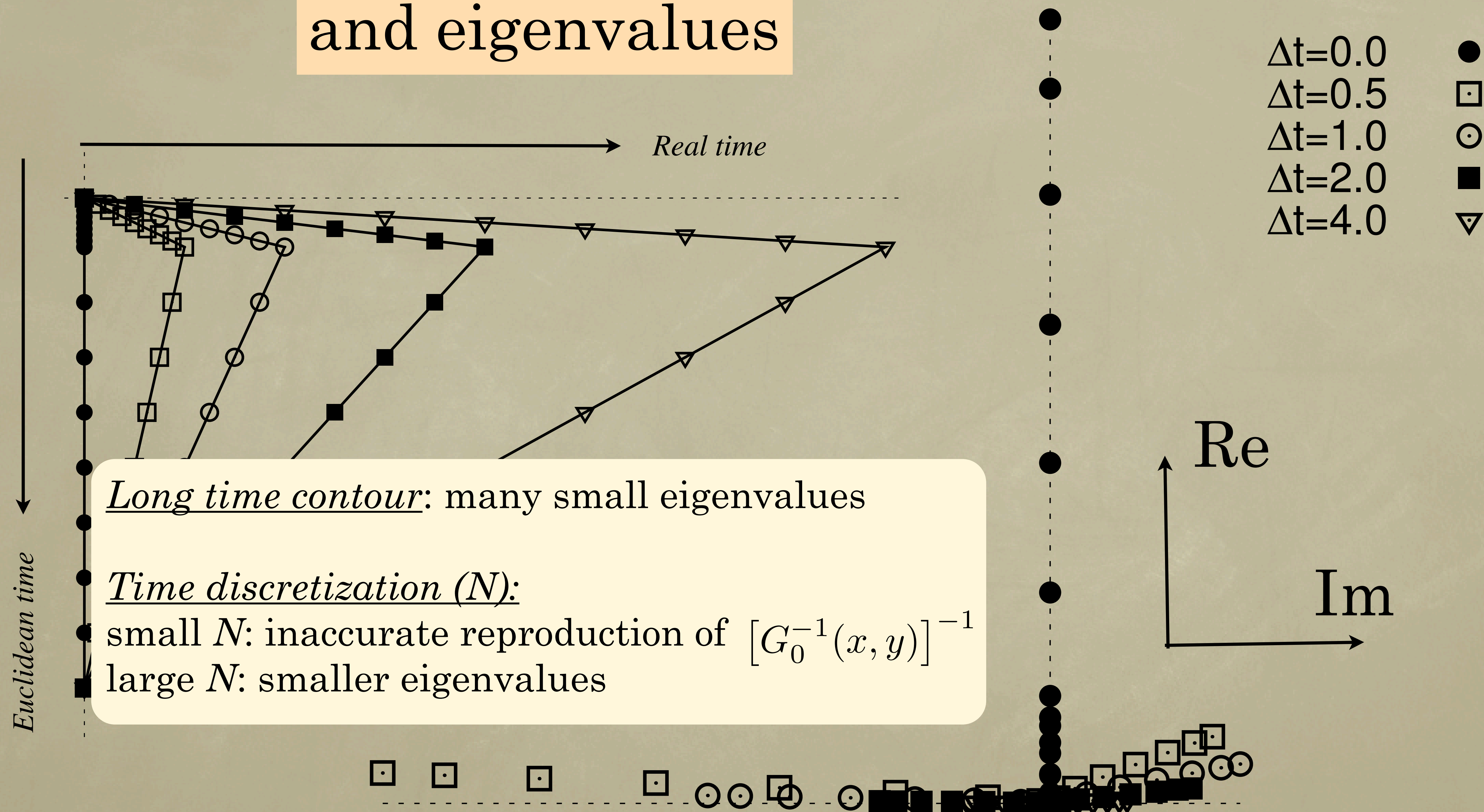


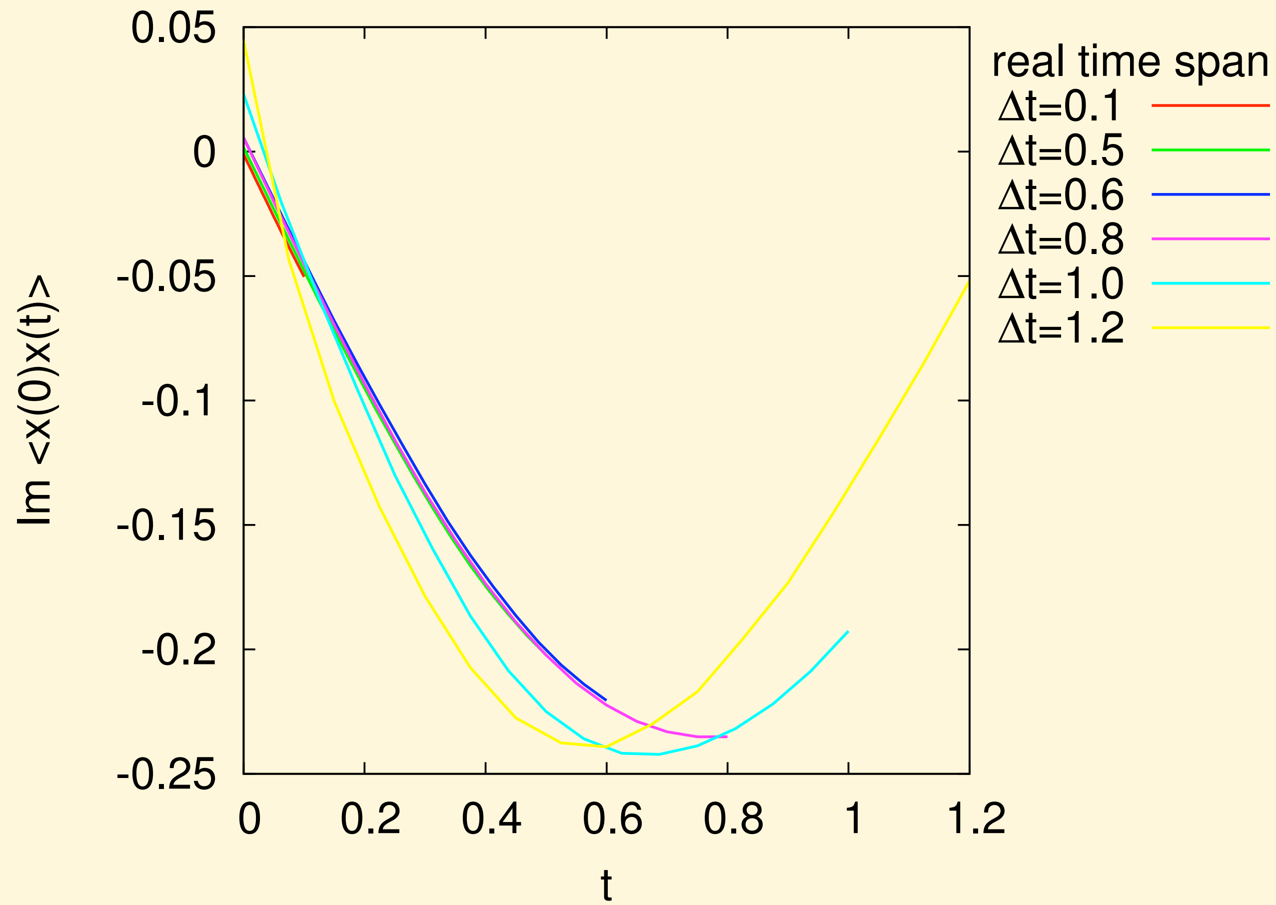
__Long time contour__: many small eigenvalues

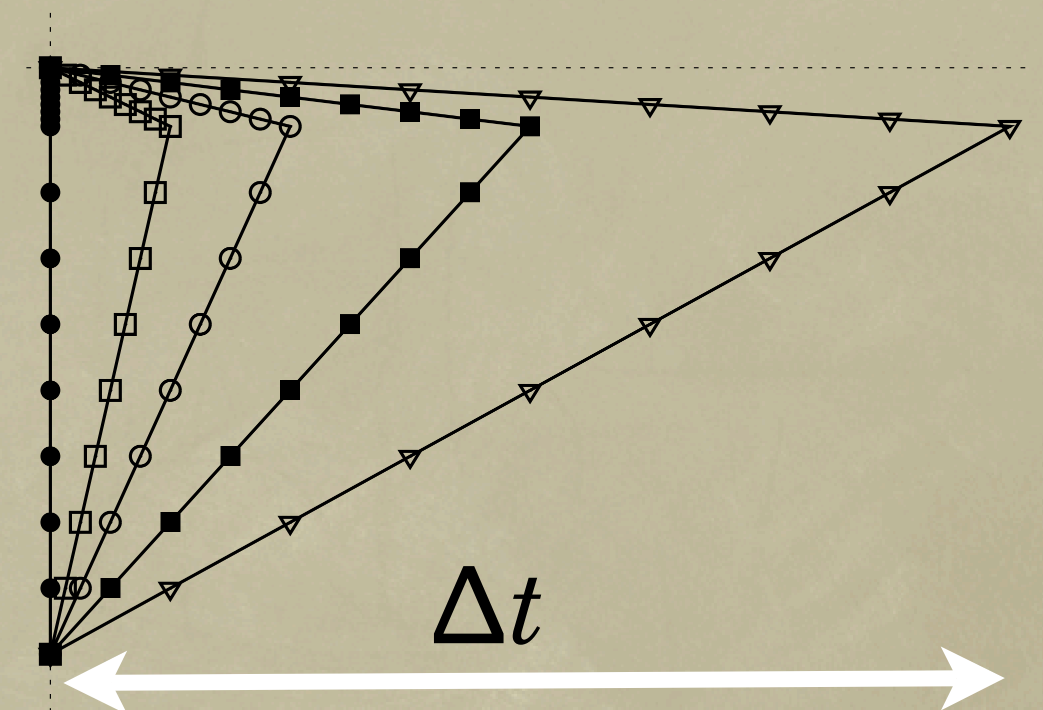
__Time discretization (N):__
small N: inaccurate reproduction of $\left[G_0^{-1}(x,y)\right]^{-1}$
large N: smaller eigenvalues

The real part of the eigenvalues are always positive if there is a tilt, no matter how small

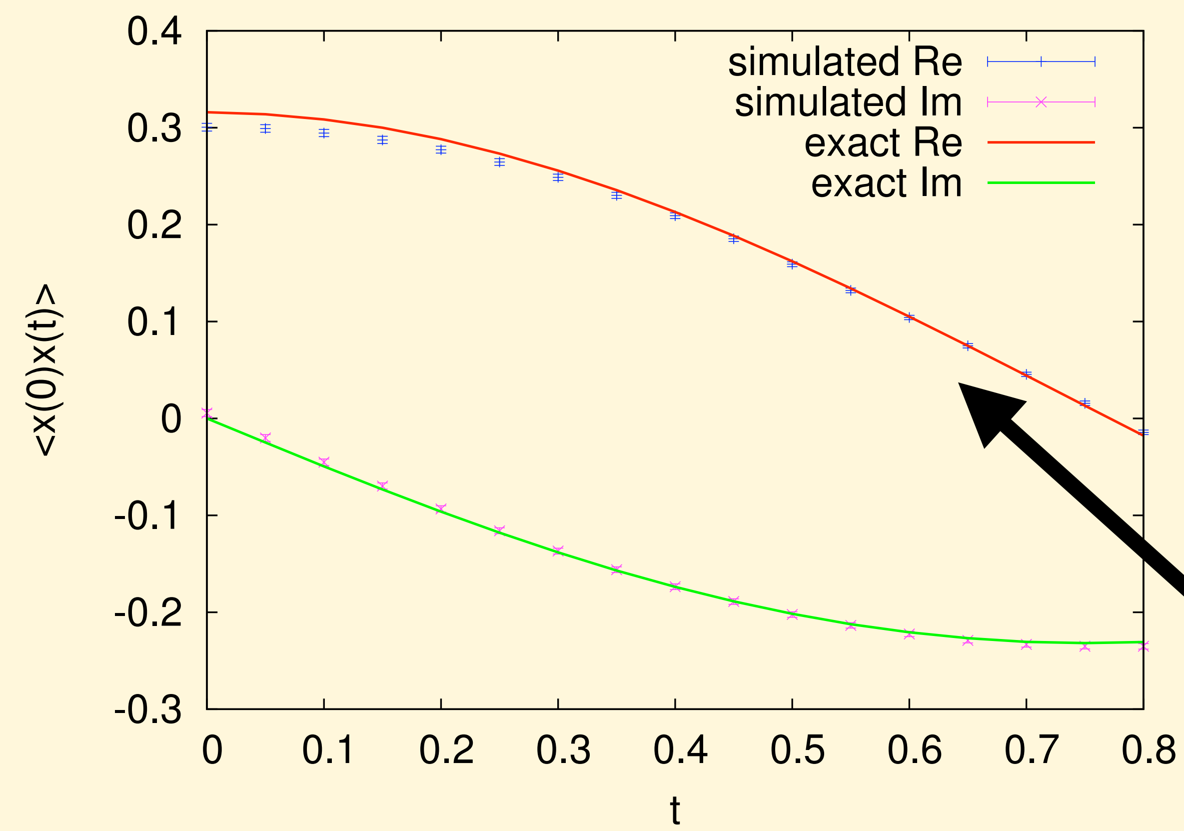
Short time intervals

$$\langle \phi(0)\phi(t) \rangle \qquad t = 0 \ldots \Delta t$$



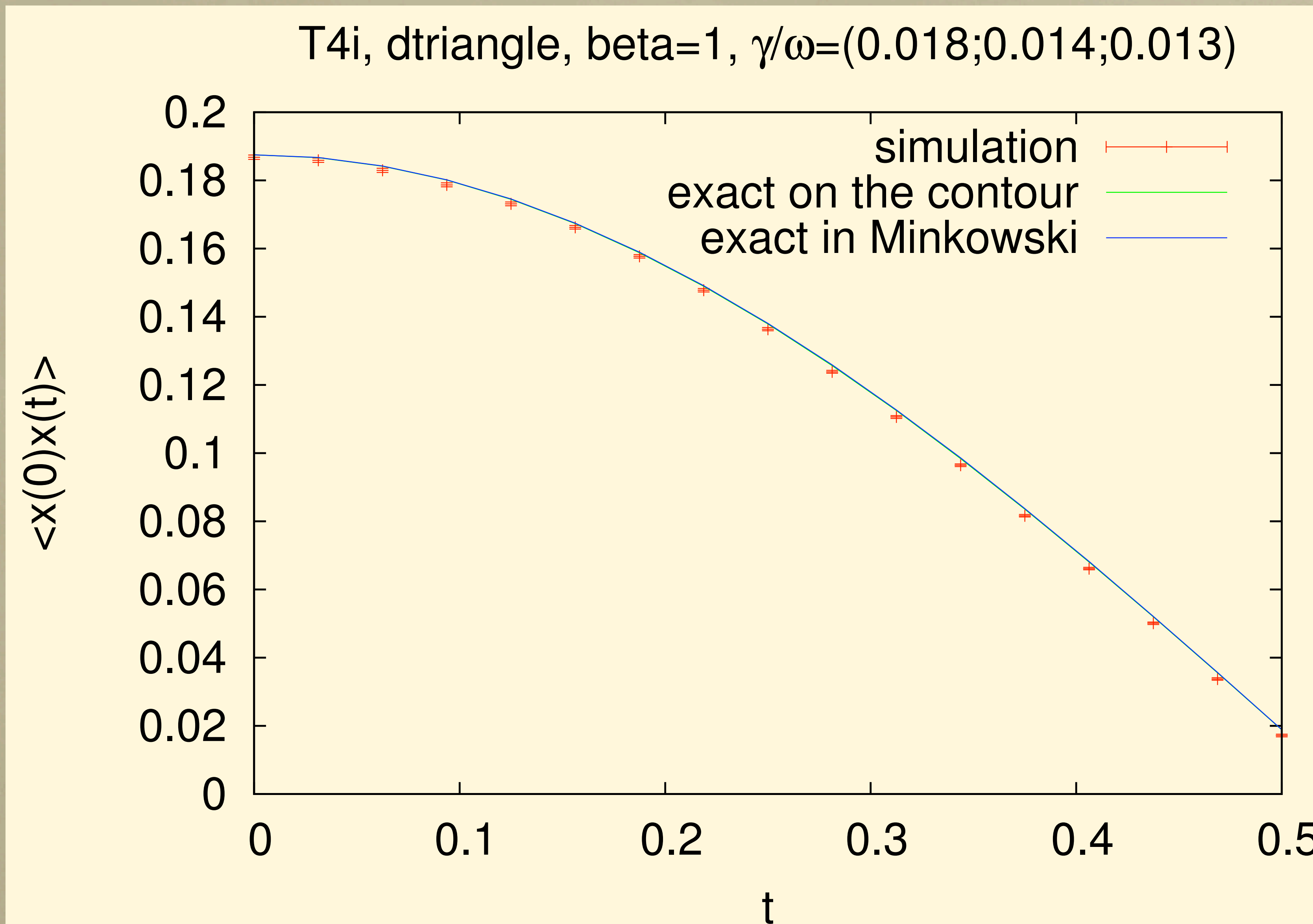


Results independent of real-time span of the grid until a certain limit



Schrodinger's exact result on the contour

A precision test: Extracting damping rate

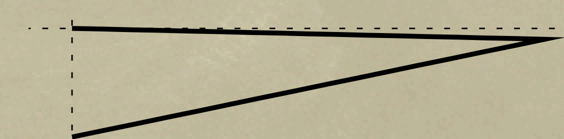


Not exactly real time:
tilt angle=0.002

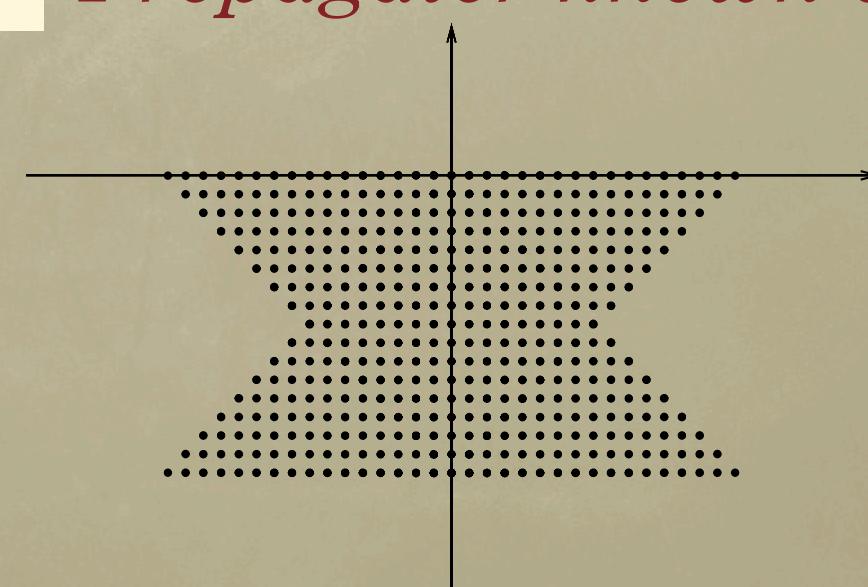
$\gamma/\omega < 2\%$ accuracy 40%

Fitting $\sim \cos(\omega t) e^{-\gamma t}$

Requires precision data



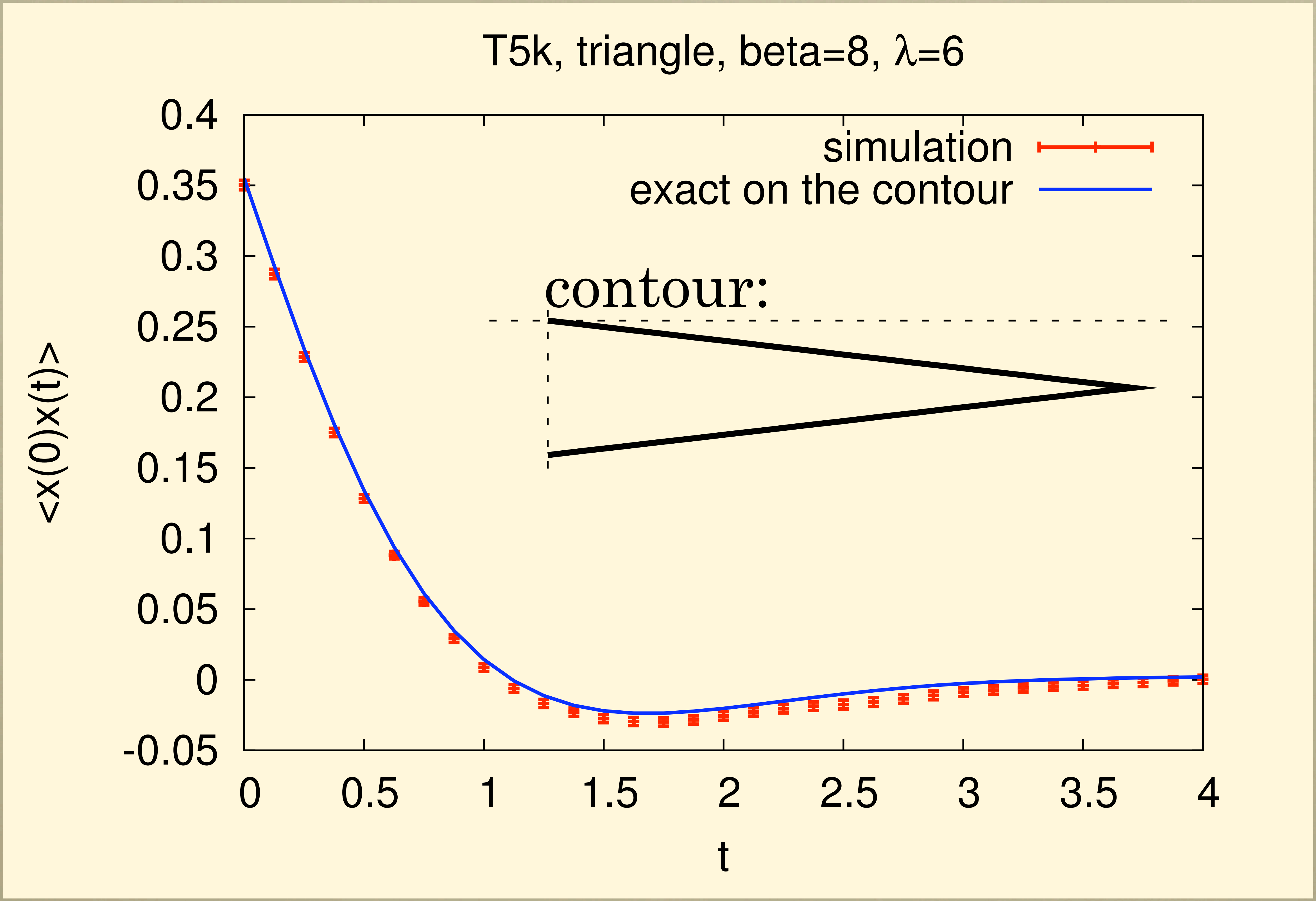
Propagator known at:



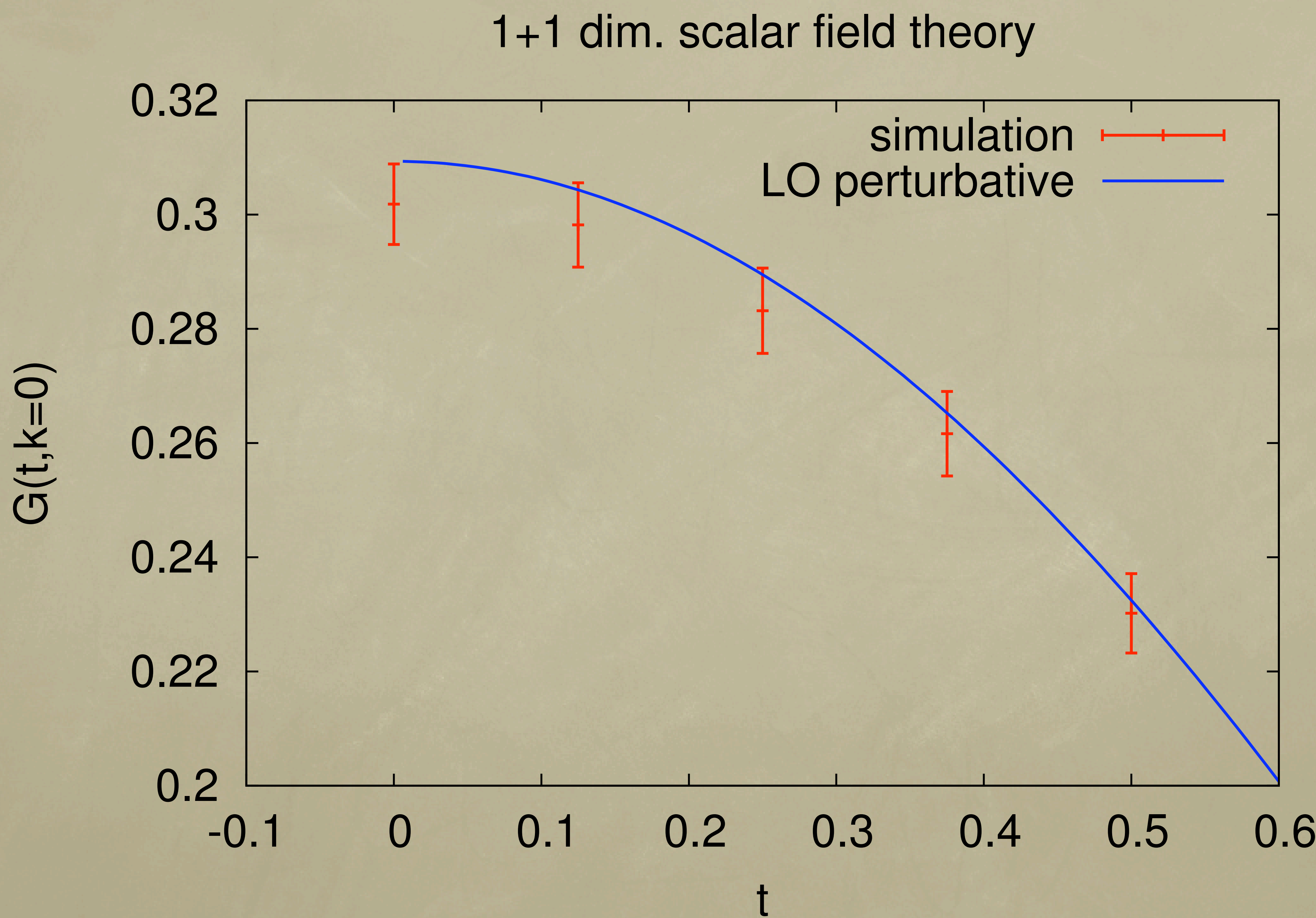
Possible improvements:

statistical analysis based on the correlation matrix

At lower temperature: longer intervals



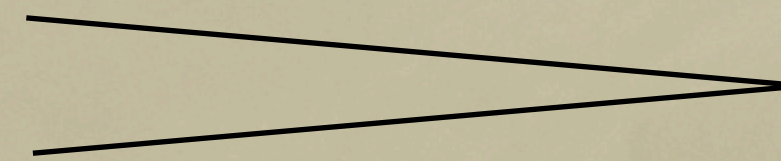
Field theory:



solid curve: Hartree approx.

Nonequilibrium: $\mathrm{Tr}\hat{\rho}\hat{\phi}e^{i\hat{H}t}\hat{\phi}e^{-i\hat{H}t}$

The contour:



$\hat{\rho}$

Tilt: regulator. (modified initial condition and more damping)

Simplest: *Gaussian density operator*

$$
\begin{aligned}
S'[\phi_+,\phi_-] &= S[\phi_+]-S[\phi_-]+\frac{1}{a_t}S_0(\phi_+(t_i),\phi_-(t_i)] \\
S_0[\phi_+,\phi_-] &= i\dot{\Phi}(\phi_+-\phi_-)-\frac{\sigma^2+1}{8\xi^2}\left((\phi_+-\Phi)^2+(\phi_--\Phi)^2\right) \\
&+ \frac{i\eta}{2\xi}\left((\phi_+-\Phi)^2-(\phi_--\Phi)^2\right) \\
&+ \frac{\sigma^2-1}{4\xi^2}(\phi_+-\Phi)(\phi_--\Phi)
\end{aligned}
$$

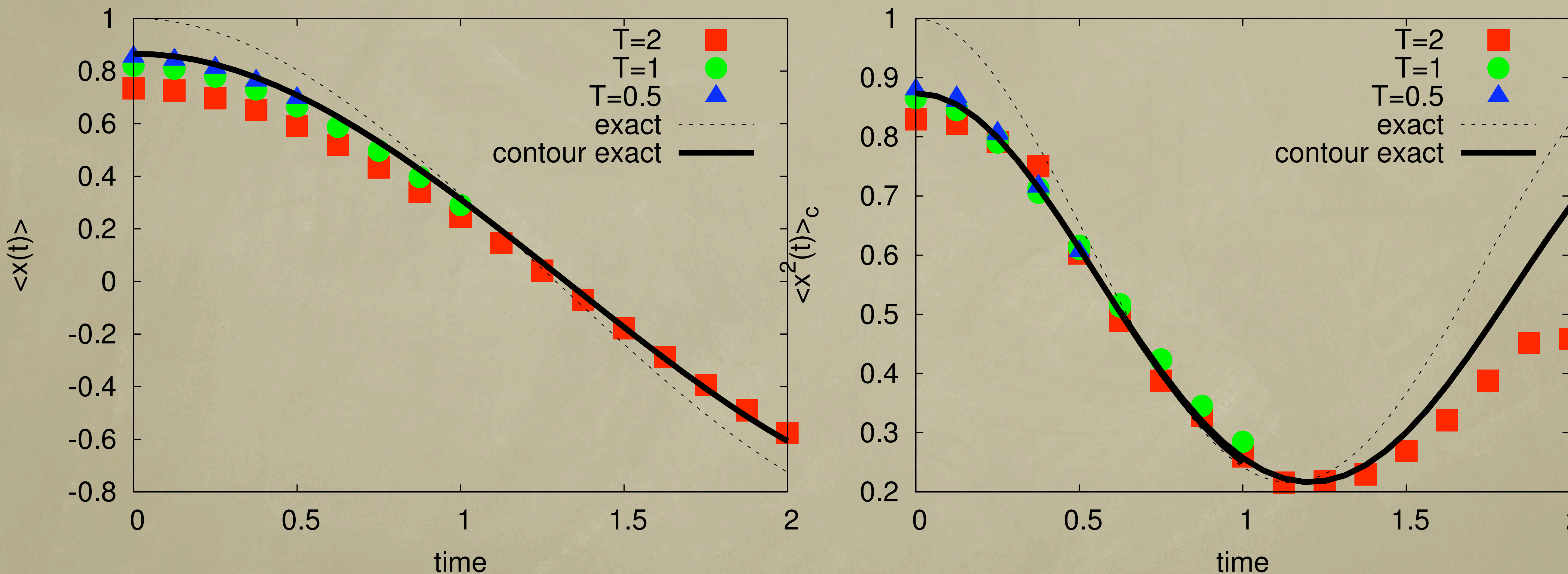
$$
\begin{aligned}
\Phi &= \langle\phi(t_i)\rangle \\
\dot{\Phi} &= \left\langle\dot{\phi}(t_i)\right\rangle \\
\xi^2 &= \langle\phi(t_i)\phi(t_i)\rangle_c \\
\eta\xi &= \frac{1}{2}\left\langle\dot{\phi}(t_i)\phi(t_i)+\phi(t_i)\dot{\phi}(t_i)\right\rangle_c \\
\eta^2+\frac{\sigma^2}{4\xi^2} &= \langle\phi(t_i)\phi(t_i)\rangle_c
\end{aligned}
$$

Field theory:

Fourier transformation in each Langevin time step

In practice:

$\langle x(t) \rangle$ $\langle x(t)x(t) \rangle_c$



Reliable for short intervals

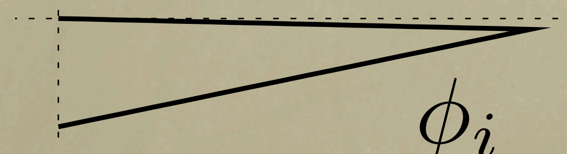
Stochastic quantization:

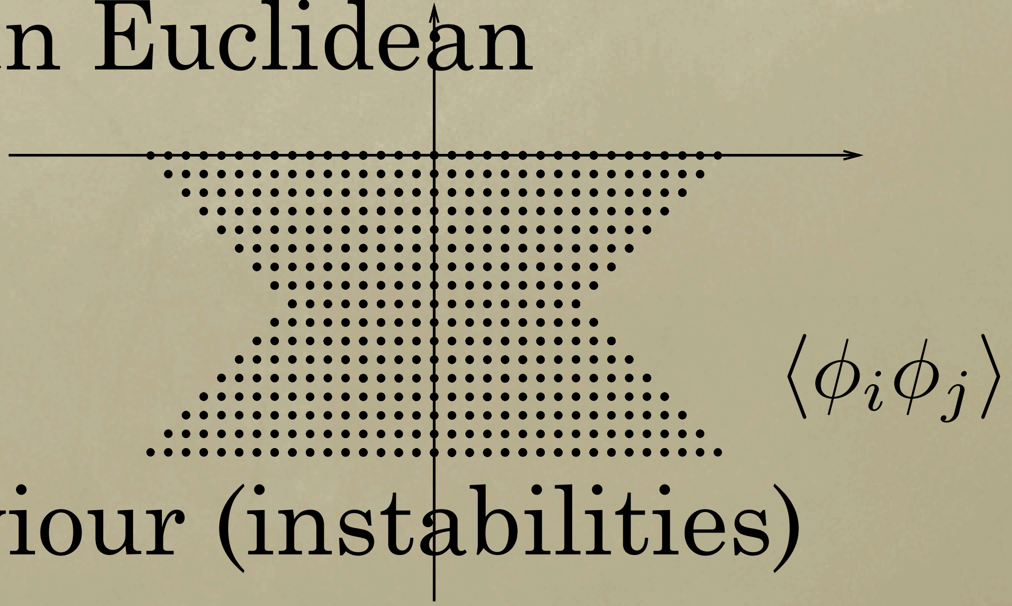
real time simulations are indeed <u>possible</u>,

- Direct simulation on short intervals so far

(longer at low temperature or weak coupling)

- For statistical estimate of real time correlators: more relevant information than Euclidean simulation may provide.





- Early nonequilibrium behaviour (instabilities)